Extending Various Tri-Ideals In Ternary Semiring To Fuzzy Setting

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***Abstract*—Various categories of fuzzy tri-ideals within ternary semirings have been introduced and examined. This study explores the properties of fuzzy tri-ideals, fuzzy tri quasi-ideals, and fuzzy bi quasi-ideals using a new approach termed 'metatheorem,' initially proposed by Tom Head [3]. It has been demonstrated that the classes of fuzzy tri-ideals, fuzzy tri quasi-ideals, and fuzzy bi quasi-ideals in ternary semirings exhibit the closure under projection property. The principal aim of employing this metatheorem is to generalize classical results into their fuzzy counterparts. Additionally, the metatheorem facilitates alternative, non-computational proofs of various results. These proofs are notably more concise and straightforward. The concept of fuzzy m-tri-ideals is further explored, providing characterizations of these ideals in relation to fuzzy tri-ideals.1. Introduction**

Keywords*—*Ternary semiring; fuzzy tri-ideal; fuzzy tri-quasi-ideal; fuzzy bi-quasi-ideals; metatheorem; projection closed.

1. Introduction

D. H. Lehmer initiated the exploration of ternary algebraic structures, known as triplexes, in 1932 [9]. The notion of semirings followed shortly thereafter, with Vandiver introducing it in 1934 [14]. By 1962, Hestenes [3] had expanded the study of ternary algebras, applying them to matrices and linear transformations. In 2005, Kar [5] defined bi-ideals and quasi-ideals specifically within the context of ternary semirings, and further extended this framework by characterizing fuzzy ideals in ternary semigroups [6]. Dutta and S. Kar [2] advanced the study by focusing on the properties of regular ternary semirings, while Dubey and Anuradha [1] developed the concept of prime quasi-ideals within this setting. Research by Kavi Kumar and Bin Khasim [8] provided deeper insights into fuzzy ideals and fuzzy quasi-ideals in ternary semirings. Additionally, Palani Kumar and Arulmozhi [10] made significant contributions by introducing different types of tri-ideals within these algebraic systems.

The generalization of ideals within algebraic structures has been both an essential and necessary step for advancing the study of these structures. One-sided ideals serve as a broadening of classical ideals, while quasi-ideals further extend this notion by encompassing both left and right ideals. Bi-ideals represent an expansion of quasi-ideals. Palanikumar and Arulmozhi [10] introduced new generalizations in the context of ternary semigroups, including tri-ideals, bi-quasi-ideals, tri-quasi-ideals, and quasi-interior ideals, which generalize bi-ideals, quasi-ideals, and interior ideals.

In 1995, Tom Head [3] introduced a metatheorem that provided a framework for examining the core properties of fuzzy algebraic structures. This metatheoretical approach was subsequently applied to the analysis of semigroups and semirings in later research [11, 12, 13]. The metatheorem provides various techniques to extend classical algebraic results to the fuzzy setting, serving as a systematic method for deriving fuzzy counterparts of classical theorems. Consequently, the clarity and conciseness of the proofs presented here, in comparison to existing ones, underscore the effectiveness of the metatheorem approach as a tool for investigating fuzzy algebraic structures. Additionally, the metatheorem is utilized to offer alternative, non-calculative proofs of several results.

In this paper, we extend various types of tri-ideals in ternary semirings to the fuzzy context through the application of the metatheorem. Originally formulated by Tom Head [3], the metatheorem provides a framework for deriving fuzzy analogs of classical results. It is demonstrated that the different classes of fuzzy ideals remain projection closed.

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2. Preliminaries

For the definitions and basic results of ternary semirings we refer [1, 2, 5]. Zadeh [15] defined fuzzy set λ on a set X as a mapping $\lambda : X \to [0,1]$. Let (S, +, $\circ$) be ternary semiring. Let $F(S)(C_{SS})$ denotes the set of all fuzzy subsets (fuzzy subsemirings of ternary semiring S.

Definition 2.1. $\lambda \in F(S)$ is called a *fuzzy subsemiring* of S if $\gamma(x+y) \geq \min\{\gamma(x), \gamma(y)\}$ and $\gamma(xyz) \geq \min \min\{\lambda(x), \lambda(y), \lambda(z)\} \forall x, y, z \in S.$

Definition 2.2. $\lambda \in F(S)$ is called a

(1) *fuzzy left (lateral, right) ideal* of S if $\gamma(x+y)\geq$ and $\gamma(xyz)\geq\gamma(z)$ ($\gamma(xyz)\geq\gamma(y),\gamma(xyz)\geq\gamma(x)$) $\forall\, x,\ y,z\in S$.

(2) *fuzzy ideal* of S if $\gamma(x+y)\geq\gamma(xy)\geq\gamma(y)$ and $\gamma(xyz)\geq\gamma(y)$, $\gamma(xyz)\geq\gamma(y)$, and $\gamma(xyz)\geq\gamma(x)$ $\forall\, x, y,z\in S$.

Definition 2.3. [6] $\lambda\in C_{SS}$ is called a *fuzzy bi-ideal* (*fuzzy interior ideal*) of S if $\lambda\circ S\circ\lambda\circ S\circ\lambda\subseteq\lambda$ ($S\circ\lambda\circ S\circ\lambda\circ S\subseteq\lambda$).

Definition 2.4. [6] $\lambda\in F(S)$ is called a *fuzzy quasi-ideal* of S if $(\lambda\circ S\circ S)\cap(S\circ\lambda\circ S\cup S\circ S\circ\lambda\circ S\circ S)$ $\cap(S\circ S\circ\lambda)\subseteq\lambda$.

Now we briefly study 'metatheorem' formulated by Tom Head [3] in the year 1995 for a ternary semiring S. Then the mapping $Chi: P(S)\to C(S)$ defined by $Chi(A)=\chi_A$ is a bijection.

Proposition 2.5. $P(S)\cong C(S)$ under the isomorphism *Chi*.

Definition 2.6 [3] Consider a ternary semiring S, $\lambda\in F(S)$ and $r\in J=[0,1)$. Then the function Rep : $F(S)\to C(S)^J$ is defined by

$$\text{Rep}(\lambda)(r)(x)=\begin{cases}1 & \text{if } \lambda(x)>r\\ 0 & \text{if } \lambda(x)\leq r\end{cases}$$

Proposition 2.7 [3] The function Rep is injective and $\text{Rep}\left(\bigcap_{i=1}^{k}\lambda_i\right)=\bigcap_{i=1}^{k}\text{Rep}(\lambda_i)$ and $\text{Rep}\left(\bigcup_{i\in M}\lambda_i\right)=\bigcup_{i\in M}\text{Rep}(\lambda_i)$, where $\mu_i\in F(S)$ and M is an index set.

Proposition 2.8. [3] Rep serves as an order isomorphism of $F(S)$ onto its image $I(S)$.

We introduce two binary operations +, $\circ$ on S as follows:

(1) $+: F(S)\times F(S)\times F(S)\to F(S)$ is defined by:

$$(\lambda_1+\lambda_2+\lambda_3)(x)=\begin{cases}\sup\limits_{x=x_1+x_2+x_3}[\min\{(\lambda_1(x_1),\lambda_2(x_2),\lambda_3(x_3)\}]\\ 0 \quad \text{if } x \text{ not expressed as } x=x_1+x_2+x_3\end{cases}$$

and

(2) $\circ: F(S)\times F(S)\times F(S)\to F(S)$ is defined by:

$$(\lambda_1\circ\lambda_2\circ\lambda_3)(x)=\begin{cases}\sup\limits_{x=x_1\circ x_2\circ x_3}[\min\{(\lambda_1(x_1),\lambda_2(x_2),\lambda_3(x_3)\}]\\ 0 \quad \text{if } x \text{ not expressed as } x=x_1\circ x_2\circ x_3\end{cases}$$

Proposition 2.9. [3] Let S be a ternary semiring. Then $\text{Rep}(\lambda_1+\lambda_2)=\text{Rep}(\lambda_1)+\text{Rep}(\lambda_2)$ and $\text{Rep}(\lambda_1\circ\lambda_2)=\text{Rep}(\lambda_1)\circ\text{Rep}(\lambda_2)$ for $\lambda_1,\lambda_2\in F(S)$.

Proposition 2.10. [3] Consider C as a collection of fuzzy sets within a semiring S. We define C as being projection closed if, for every $\lambda\in C$ and for any $r\in J$, the projection $\text{Rep}(\lambda)(r)$ remains an element of C.

Proposition 2.11. [1] $D_1,D_2(\mathcal{D}_1,\mathcal{D}_2)$ be the classes of crisp (fuzzy) subsets of a semiring S. Then, $\mathcal{D}_1\subseteq\mathcal{D}_2(\mathcal{D}_1=\mathcal{D}_2)\Leftrightarrow D_1\subseteq D_2(D_1=D_2)$.

Proposition 2.12. [3] $(S,+,\circ)$ be a ternary semiring. Consider the algebra-$(F(S)$, inf, sup, + , $\circ)$. Let $L(d_1,d_2,...,d_m)$ and $M(d_1,d_2,...,d_m)$ be two expression defined over the variables set $\{d_1,d_2,...,d_m\}$ and operations set { inf, sup, + , $\circ$} on $P(S)$. Let $\mathcal{D}_1,\mathcal{D}_2,...,\mathcal{D}_m$ be projection closed classes of fuzzy sets S and $D_1,D_2,...,D_m$ be their respective crisps classes . Then

$$L(\lambda_1,\lambda_2,...,\lambda_m)\ \text{REL}\ M(\lambda_1,\lambda_2,...,\lambda_m)$$

holds $\forall\ \lambda_1$ *in* $\mathcal{D}_1,...,\lambda_m$ *in* $\mathcal{D}_m\Leftrightarrow$ it holds $\forall\ \lambda_1$ *in* $D_1,...,\lambda_m$ *in* D_m, where REL represent one of the three symbols $\leq$, = or $\geq$.

3. Various Fuzzy Tri-ideals in Ternary Semirings

Definition 3.1. $\lambda\in C_{SS}$ is called a

(1) *fuzzy left tri quasi-ideal* of S if $(S\circ S\circ\lambda)\cap(\lambda\circ\lambda\circ S\circ S\circ\lambda\circ\lambda\circ\lambda)\subseteq\lambda$..

(2) *fuzzy lateral tri quasi-ideal* of S if $(S\circ\lambda\circ S)\cap(\lambda\circ\lambda\circ S\circ\lambda\circ S\circ\lambda\circ\lambda)\subseteq\lambda$.

(3) *fuzzy right tri quasi-ideal* of S if $(\lambda\circ S\circ S)\cap(\lambda\circ\lambda\circ\lambda\circ S\circ S\circ\lambda\circ\lambda)\subseteq\lambda$.

(4) *fuzzy tri quasi-ideal* of S if λ is a fuzzy left tri-quasi-ideal, fuzzy lateral tri-quasi-ideal and fuzzy right tri-quasi-ideal of S.

Definition 3.2. $\lambda\in C_{SS}$ is called a

(1) *fuzzy left bi-quasi-ideal* of S if $(S\circ S\circ\lambda)\cap(\lambda\circ S\circ\lambda\circ S\circ\lambda)\subseteq\lambda$.

(2) *fuzzy lateral bi-quasi-ideal* of S if $(S\circ\lambda\circ S)\cap(\lambda\circ S\circ\lambda\circ S\circ\lambda)\subseteq\lambda$.

(3) *fuzzy right bi-quasi-ideal* of S if $(\lambda\circ S\circ S)\cap(\lambda\circ S\circ\lambda\circ S\circ\lambda)\subseteq\lambda$.

(4) *fuzzy bi-quasi-ideal* of S if λ is a fuzzy left bi-quasi-ideal, fuzzy lateral bi-quasi-ideal and fuzzy right bi-quasi-ideal of S.

Definition 3.1. $\lambda\in C_{ss}$ is called a

(1) *fuzzy left tri-ideal* of S if $\lambda\circ\lambda\circ S\circ S\circ\lambda\circ\lambda\circ\lambda\subseteq\lambda$.

(2) *fuzzy lateral tri-ideal* of S if $\lambda\circ\lambda\circ S\circ\lambda\circ S\circ\lambda\circ\lambda\subseteq\lambda$.

(3) *fuzzy right tri-ideal* of S if $\lambda\circ\lambda\circ\lambda\circ S\circ S\circ\lambda\circ\lambda\subseteq\lambda$.

(4) *fuzzy tri -ideal* of S if λ is a fuzzy left tri-ideal, fuzzy lateral tri-ideal and fuzzy right tri-ideal of S.

Theorem 3.4. C_{ss} is projection closed.

Proof. Let $\lambda\in C_{ss}$. Then, $\lambda(x+y)\geq\min\{\lambda(x),\lambda(y)\}$ and $\lambda(xyz)\geq\min\{\lambda(x),\lambda(y),\lambda(z)\}$ $\forall$ x,y,z $\in$ S. To show that C_{ss} is projection closed, we establish Rep(λ)(r)(x+y) $\geq$ min{Rep(λ)(r)(x), Rep(λ)(r)(y)} and Rep(λ)(r)(xyz) $\geq$ min{Rep(λ)(r)(x), Rep(λ)(r)(y), Rep(λ)(r)(z)} $\forall$ x,y,z $\in$ S and $\forall$ r $\in J$.

Rep(λ)(r)(x+y) $\geq$ min{Rep(λ)(r)(x), Rep(λ)(r)(y)} $\forall$ x,y $\in$ S and $\forall$ r $\in J$ follows from Theorem 3.4 of [11] and Rep(δ)(r)(xyz) $\geq$ min{Rep(δ)(r)(x), Rep(δ)(r)(y), $\forall$ x,y,z $\in$ S and $\forall$ r $\in J$ follows from Theorem 4.1 of [14].

Theorem 3.5. C_l, C_{lt}, C_r, and C_i are projection closed.

Theorem 3.6. C_b is projection closed.

Proof. Let $\lambda\in C_b$. Then $\lambda\in C_{ss}$ and $\lambda\circ S\circ\lambda\circ S\circ\lambda\subseteq\lambda$. Utilizing Theorem 3.4, it is required to show that $\mathrm{Rep}(\lambda)(r)\in C_b$ $\forall\, r\in J$. For $r\in J$, we have,

$\mathrm{Rep}(\lambda)(r)\geq\mathrm{Rep}(\lambda\circ S\circ\lambda\circ S\circ\lambda)(r)$ by Proposition 2.8

$=\mathrm{Rep}(\lambda)(r)\circ\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(\lambda)(r)\circ\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(\lambda)(r)$ by Proposition 2.9.

$=\mathrm{Rep}(\lambda)(r)\circ S\circ\mathrm{Rep}(\lambda)(r)\circ S\circ\mathrm{Rep}(\lambda)(r)$.

Thus, $\mathrm{Rep}(\lambda)(r)\in C_b$ $\forall\, r\in J$.

Theorem 3.7. C_{in} is projection closed.

Theorem 3.8. C_q is projection closed.

Proof. Let $\lambda\in C_q$. Then λ is an additive subsemi-group of S and $\lambda\circ S\circ S\cap(S\circ\lambda\circ S\cup S\circ S\circ\lambda\circ S\circ S)\cap S\circ S\circ\lambda\subseteq\lambda$. Utilizing Theorem 3.4, it is require d to show that $\mathrm{Rep}(\lambda)(r)\in C_q$ $\forall\, r\in J$. For $r\in J$, we have,

$\mathrm{Rep}(\lambda)(r)\geq\mathrm{Rep}(\lambda\circ S\circ S\cap(S\circ\lambda\circ S\cup S\circ S\circ\lambda\circ S\circ S)\cap S\circ S\circ\lambda(r)$ by Proposition 2.8.

$=\mathrm{Rep}(\lambda\circ S\circ S)(r)\cap(\mathrm{Rep}(S\circ\lambda\circ S)(r)\cup\mathrm{Rep}(S\circ S\circ\lambda\circ S\circ S)(r))\cap\mathrm{Rep}(S\circ S\circ\lambda)(r)$ by Proposition 2.7.

$=\mathrm{Rep}(\lambda)(r)\circ\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(S)(r)\cap(\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(\lambda)(r)\circ\mathrm{Rep}(S)(r)\cup\mathrm{Rep}(S)(r)\circ(\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(\lambda)(r)\circ\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(S)(r))\cap\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(\lambda)(r)$ by Proposition 2.9.

$=\mathrm{Rep}(\lambda)(r)\circ S\circ S\cap(S\circ\mathrm{Rep}(\lambda)(r)\circ S\cup S\circ S\circ\mathrm{Rep}(\lambda)(r)\circ S\circ S)\cap S\circ S\circ\mathrm{Rep}(\lambda)(r)$.

$\mathrm{Rep}(\lambda)(r)\in C_q$ $\forall\, r\in J$.

Theorem 3.9. C_{bqr}, C_{bqlt}, C_{bqr}, and C_{bq} are projection closed.

Proof. Let $\lambda\in C_{bqr}$. Then $\lambda\in C_{ss}$ and $(\lambda\circ S\circ S)\circ(\lambda\circ S\circ\lambda\circ S\circ\lambda)\subseteq\lambda$. Utilizing Theorem 3.4, it is required to show that $\mathrm{Rep}(\lambda)(r)\in C_{bqr}$ $\forall\, r\in J$. For $r\in J$, we have,

$\mathrm{Rep}(\lambda)(r)\geq\mathrm{Rep}(\lambda\circ S\circ S\cap\lambda\circ S\circ\lambda\circ S\circ\lambda)(r)$ by Proposition 2.8

$=\mathrm{Rep}(\lambda\circ S\circ S)(r)\cap\mathrm{Rep}(\lambda\circ S\circ\lambda\circ S\circ\lambda)(r)$ by Proposition 2.7.

$=\mathrm{Rep}(\lambda)(r)\circ\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(S)(r)\cap\mathrm{Rep}(\lambda)(r)\circ\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(\lambda)(r)\circ\mathrm{Rep}(S)(r)\circ\mathrm{Rep}(\lambda)(r)$ by Proposition 2.9

$=\mathrm{Rep}(\lambda)(r)\circ S\circ S\cap\mathrm{Rep}(\lambda)(r)\circ S\circ\mathrm{Rep}(\lambda)(r)\circ S\circ\mathrm{Rep}(\lambda)(r)$.

Thus, $\mathrm{Rep}(\lambda)(r)\in C_{bqr}$ $\forall\, r\in J$.

Theorem 3.10. C_{tql}, C_{tqlt}, C_{tqr}, and C_{tq} are projection closed.

Proof. Let $\lambda\in C_{tql}$. Then $\lambda\in C_{ss}$ and $(S\circ S\circ\lambda)\cap(\lambda\circ\lambda\circ S\circ S\circ\lambda\circ\lambda\circ\lambda)\subseteq\lambda$. Utilizing

Theorem 3.4, it is required to show that $\text{Rep}(\lambda)(\text{r}) \in C_{tql}$ $\forall\ r \in J$. For $r \in J$, we have,

$\text{Rep}(\lambda)(r) \geq$ $\text{Rep}(S \circ S \circ \lambda \cap \lambda \circ \lambda \circ S \circ S \circ \lambda \circ \lambda \circ \lambda)(r)$ by Proposition 2.8

$=\text{Rep}(S \circ S \circ \lambda)(r) \cap \text{Rep}(\lambda \circ \lambda \circ S \circ S \circ \lambda \circ \lambda \circ \lambda)(r)$ by Proposition 2.7.

$=\text{Rep}(\text{S})(r) \circ \text{Rep}(\text{S})(r) \circ \text{Rep}(\lambda)(r) \cap (\text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r) \circ$

$\text{Rep}(\text{S})(r) \circ \text{Rep}(\text{S})(r) \circ \text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda))(r) \circ \text{Rep}(\lambda))(r)$ by Proposition 2.9

$=S \circ S \circ \text{Rep}(\lambda)(r) \cap \text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r) \circ S \circ S \circ \text{Rep}(\lambda)(r)$

$\circ \text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r)$.

Thus $\text{Rep}(\lambda)(r) \in C_{tql}$ $\forall\ r \in J$.

Theorem 3.11. $C_{tilt}, C_{til}, C_{tir}$, and C_{ti} are projection closed.

Proof. Let $\lambda \in C_{tilt}$. Then $\lambda \in C_{ss}$ and $\lambda \circ \lambda \circ S \circ \lambda \circ S \circ \lambda \circ \lambda \subseteq \lambda$. Utilizing Theorem 3.4, it is required to show that $\text{Rep}(\lambda)(\text{r}) \in C_{tilt}$ $\forall\ r \in J$. For $r \in J$, we have,

$\text{Rep}(\lambda)(r) \geq$ $\text{Rep}(\lambda \circ \lambda \circ S \circ \lambda \circ S \circ \lambda \circ \lambda)(r)$ by Proposition 2.8

$=\text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r) \circ \text{Rep}(\text{S})(r) \cap (\text{Rep}(\lambda)(r) \circ$

$\text{Rep}(\text{S})(r) \circ \text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r)$ by Proposition 2.9

$=\text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r) \circ S \circ \text{Rep}(\lambda)(r) \circ S \circ \text{Rep}(\lambda)(r)$

$\circ \text{Rep}(\lambda)(r)$.

Thus, $\text{Rep}(\lambda)(r) \in C_{tilt}$ $\forall\ r \in J$.

4. Fuzzy Tri-ideals of a ternary Semirings

Theorem 4.1. The following statements hold in a ternary semiring S:

(1) Every fuzzy left (lateral, right) ideal of S is also a fuzzy left (lateral, right) tri-ideal of S.

(2) Every fuzzy ideal of S is also a fuzzy tri-ideal of S.

(3) Every fuzzy left (right) bi-quasi-ideal of S is also a fuzzy left(right) tri-quasi-ideal of S.

(4) Every fuzzy lateral bi-quasi-ideal of S is also a fuzzy lateral tri-quasi-ideal of S.

Proof. (4) Both the classes, C_{bqlt} and C_{tqlt} of fuzzy lateral bi-quasi-ideals and fuzzy lateral tri-quasi-ideals are projection closed by Theorem 3.9 and 3.10. Therefore, $C_{bqlt} \subseteq C_{tqlt}$ if and only if $C_{bqlt} \subseteq C_{tqlt}$ follows from Proposition 2.11. By Theorem 3.3 of [10], every lateral bi-quasi-ideal of a ternary `semiring S is a lateral tri-quasi-ideals of S and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{bqlt} \subseteq C_{tqlt}$. Hence $C_{bqlt} \subseteq C_{tqlt}$.

Analogously the remaining results extends Theorem 3.1, 3.3 of [10] to fuzzy setting.

Theorem. 4.2. The following statements hold in a ternary semiring S:

(1) Every fuzzy left (lateral, right) bi-quasi-ideal of S is also a fuzzy tri-ideal of S.

(2) Every fuzzy bi-quasi-ideal of S is also a fuzzy tri-ideal of S.

(3) Every fuzzy bi-ideal (interior ideal) of S is also a fuzzy left (lateral, right) tri-ideal of S.

Proof. We establish (3) both the classes C_{in} and C_{til} are projection closed by Theorem 3.7, 3.11. Therefore, $C_{in} \subseteq C_{til} \Leftrightarrow C_{in} \subseteq C_{til}$ by Proposition 2.11, where C_{in} and C_{til} are crisp classes of interior and left tri-ideals of S. By Theorem 3.6 of [10], in a ternary semiring S, every interior ideal is a left tri-ideals and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{in} \subseteq C_{til}$. Hence $C_{in} \subseteq C_{til}$.

Analogously, the remaining results extends Theorem 3.4, 3.5 and Corollary 3.1 of [10] to fuzzy setting.

Theorem 4.3 Let $h \in C_{ss}$. Then $h \in C_{ti}$ if $\lambda \circ \mu \circ \delta \subseteq h \subseteq \lambda \cap \mu \cap \delta$ for some $\lambda \in C_r$, $\mu \in C_{lt}$ and $\delta \in C_l$.

Proof. We define the classes C = $\{\chi_H \in C(S)$ where H is a subsemiring of $S: ABC \subseteq H \subseteq A \cap B \cap C$ for some left ideal A, lateral ideal B and right ideal C of S$\}$ and $C = \{h \in C_{ss} : \alpha \circ \mu \circ \delta \subseteq h \subseteq \alpha \cap \mu \cap \delta$ for some $\lambda \in C_r$ $\mu \in C_b$ and $\delta \in C_r\}$. To show C is projection closed, let $h \in C$. Therefore, $h \in C_{ss}$ such that $\lambda \circ \mu \circ \delta \subseteq h \subseteq \lambda \cap \mu \cap \delta$ for some $\lambda \in C_r$, $\mu \in C_{lt}$ and $\mu \in C_l$. By Proposition 2.8, for $r \in J$, $\text{Rep}\{\alpha \circ \mu \circ \delta)(r) \leq \text{Rep}(h)(r) \leq \text{Rep}(a \cap \mu \cap \delta)(r)$ for $r \in J$. By Proposition 2.7 and 2.9, we have, for $r \in J$, $\text{Rep}(\alpha)(r) \circ$ $\text{Rep}(\mu)(r) \circ$ $\text{Rep}(\delta)(r) \leq$ $\text{Rep}(h)(r) \leq$ $\text{Rep}(\alpha)(r) \cap$ $\text{Rep}(\mu)(r) \cap$ $\text{Rep}(\delta)(r)$ .Since C_l, C_{lt} and C_r are projection closed by Theorem 3.5, we have, $\text{Rep}(\lambda)(r) \in C_r$, $\text{Rep}(\mu)(r) \in C_{lt}$ and $\text{Rep}(\delta)(r) \in C_l$. Hence $\text{Rep}(h)(r) \in C$ for some $\forall\ r \in J$. Hence C is Projection closed. Also C_{ti} is projection closed. Therefore, by Proposition 2.11, D $C \subseteq C_{lt} \Leftrightarrow C \subseteq C_{lt}$.

The later proposition follows as a ternary semiring H of a ternary semiring S is a tri-ideal if $A_1B_1C_1 \subseteq H \subseteq A_1 \cap B_1 \cap C_1$ for some left ideal A_1, lateral ideal B_1 and right ideal C_1 of S by Theorem 3.7 of [10] and $P(S) \cong C(S)$ under the isomorphism *Chi*.

Theorem 4.4. If $\mu \in C_l, \gamma \in C_{lt}$ and $\delta \in C_r$. Then

(1) $\mu \cap \gamma \cap \delta \in C_{ti}$.

(2) $\mu \circ \gamma \circ \delta \in C_{ti}$.

Proof. To establish (1), define the classes $C_{r,lt,i} = \{ f_1 \cap f_2 \cap f_3 : f_1 \in C_l,\ f_2 \in C_{lt},\ \mu_3 \in C_r \}$ and $C_{r,lt,i} = \{\lambda \cap \mu \cap \delta : \lambda \in C_l, \mu \in C_{lt}, \delta \in C_r\}$. To show $C_{r,lt,i}$ is projection closed, let $\mu \cap \gamma \cap \delta \in C_{r,lt,i}$. For all $\lambda \in C_l, \mu \in C_{lt}, \delta \in C_r$, $\text{Rep}(\lambda \cap \mu \cap \delta)(r) = \text{Rep}(\lambda)(r) \cap \text{Rep}(\mu)(r) \cap \text{Rep}(\delta)(r)\ \forall r \in J$ by Proposition 2.8. Since C_l, C_{lt} and C_r are projection closed, we have, $\text{Rep}(\lambda)(r) \in C_l$, $\text{Rep}(\mu)(r) \in C_{lt}$ and $\text{Rep}(\delta)(r) \in C_r$ Hence $\text{Rep}(\lambda \cap \mu \cap \delta)(r) \in C_{r,lt,l}$ $\forall r \in J$. Therefore $C_{r,lt,l}$ is projection closed. Also, C_{ti} is projection closed. Therefore, by Proposition 2.11, $C_{r,lt,l} \subseteq C_{ti} \Leftrightarrow C_{r,lt,l} \subseteq C_{ti}$. By Theorem 3.4 of [10], the intersection of a right ideal, a lateral ideal and a left ideal of a ternary semiring is a tri-ideal of S and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{r,lt,l} \subseteq C_{ti}$. Hence, $C_{r,lt,l} \subseteq C_{ti}$

To establish (2), define $C_{r,lt,i} = \{ f_1 \circ f_2 \circ f_3 : f_1 \in C_l,\ f_2 \in C_{lt},\ f_3 \in C_r \}$ and $C_{r,lt,j} = \{\lambda \circ \mu \circ \delta : \lambda \in C_l, \mu \in C_{lt},\ \delta \in C_r\}$. To show $C_{l,lt,r}$ is projection closed, let $\lambda \circ \mu \circ \delta \in C_{l,lt,r}$. For all $\lambda \in C_l$, $\mu \in C_{lt}, \delta \in C_r$, $\text{Rep}(\lambda \circ \mu \circ \delta)(r) = \text{Rep}(\lambda)(r) \circ \text{Rep}(\mu)(r) \circ \text{Rep}(\delta)(r)\ \forall r \in J$ by Proposition 2.9. Since C_l, C_{lt} and C_r are projection closed, we have, $\text{Rep}(\lambda)(r) \in C_l$, $\text{Rep}(\mu)(r) \in C_{lt}$ and $\text{Rep}(\delta)(r) \in C_r$. Hence $\text{Rep}(\lambda \circ \mu \circ \delta)(r) \in C_{l,lt,r}$ $\forall r \in J$. Therefore $C_{l,lt,r}$ is projection closed. Also, C_{ti} is projection closed by Theorem 4.12. Therefore, by Proposition 2.11, $C_{l,lt,r} \subseteq C_{ti} \Leftrightarrow C_{l,lt,r} \subseteq C_{ti}$. By Theorem 3.2 of [10], the product of a left ideal, a lateral ideal and a right ideal of a ternary semiring is a tri-ideal and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_{l,lt,r} \subseteq C_{ti}$. Hence, $C_{l,lt,r} \subseteq C_{ti}$.

Theorem 4.5. The following assertions hold in a ternary semiring *S*.

(1) Let $\alpha \in C_{til}(C_{tilt}, C_{tir})$ and $\gamma \in C_i$, Then, $\alpha \cap \gamma \in C_{til}(C_{tit}, C_{tir})$.

(2) $\alpha \in C_{ti}$ and $\gamma \in C_i$, then $\alpha \cap \gamma \in C_{ti}$.

(3) $\alpha \in C_{ti}$ and $\gamma \in C_{in}(C_{bq},\ C_{tq})$, The $\alpha \cap \gamma \in C_{ti}$.

Proof. To establish (1), define the classes $C_{til,i} = \{L_1 \cap I_1 : L_1 \in C_{til}, I_1 \in C_i\}$ and $C_{til,i} = \{\alpha \cap \gamma : \alpha \in\in C_{til}, \gamma \in C_i\}$ on *S*. By proceeding same as in Theorem 5.3, it can be shown easily that $C_{til,i}$ is projection closed. Since C_{til} is projection closed, therefore, $C_{til,i} \subseteq C_{til} \Leftrightarrow C_{til,i} \subseteq C_{til}$ follows from Proposition 2.11. By Theorem 3.8 of [11], the intersection of a left tri-ideal and an ideal is a left tri-ideal together with $P(S) \cong C(S)$ under the isomorphism *Chi* yields $C_{til,i} \subseteq C_{til}$. Hence, $C_{til,i} \subseteq C_{til}$.

Analogously the remaining results extends Theorem 3.8 and Corollary 3.2 of [10] to fuzzy setting.

5. Fuzzy m Tri-ideals in Ternary Semirings

Definition 5.1. Let $m \geq 1$. Then $\lambda \in C_{ss}$ is called a

(1) *m-fuzzy bi-ideal* of S if $\lambda \circ S^m \circ \lambda \circ S^m \circ \lambda \subseteq \lambda$

(2) *m-fuzzy left tri-ideal* of S if $\lambda \circ \lambda \circ S^m \circ \lambda \circ \lambda \circ \lambda \subseteq \lambda$.

(3) *m-fuzzy lateral tri-ideal* of S if $\lambda \circ \lambda \circ S^m \circ \lambda \circ S^m \circ \lambda \circ \lambda \subseteq \lambda$.

(4) *m-fuzzy right tri-ideal* of S if $\lambda \circ \lambda \circ \lambda \circ S^m \circ \lambda \circ \lambda \subseteq \lambda$.

(5) *m-fuzzy tri-ideal* of S if it is a fuzzy right m- tri-ideal, fuzzy lateral m- tri-ideal and fuzzy left m- tri-ideal of S.

Definition 5.2. Let $m \geq 1$. Then $\lambda \in F(S)$ is called *m-fuzzy quasi-ideal* of S if λ is an additive fuzzy sub- semigroup of S and $\lambda \circ S^m \cap S^m \circ \lambda \circ S^m \cap S^m \circ \lambda \subseteq \lambda$.

Note: Throughout this section

(i) C_{mb}, C_{mq} denotes the classes of m-fuzzy bi-ideals and m-fuzzy quasi-ideals of S and C_{mb}, C_{mq} be their respective crisp classes.

(ii) $C_{m,\,til}$, $C_{m,\,tilt}$, $C_{m,\,tir}$, and $C_{m,\,ti}$ denotes the classes of m-fuzzy (left, lateral, right, tri) ideals of S and $C_{m,\,til}$, $C_{m,\,tilt}$, $C_{m,\,tir}$ and $C_{m,\,ti}$ be their respective crisp classes.

Theorem 5.3. The class C_{mb} of all m-fuzzy left ideals of a ternary semiring S is projection closed.

Proof. Let $\lambda \in C_{mb}$. Then $\lambda \in C_{ss}$ and $\lambda \circ S^m \circ \lambda \circ S^m \circ \lambda \subseteq \lambda$. Utilizing Theorem 3.4, it is required to show that $\text{Rep}(\lambda)(r) \in C_{mb}\ \forall\ r \in J$. For $r \in J$, we have,

$\text{Rep}(\lambda)(r) \geq \text{Rep}(\lambda \circ S^m \circ \lambda \circ S^m \circ \lambda)(r)$

$=\text{Rep}(\lambda)(r) \circ (\text{Rep(S)}(r))^m \circ \text{Rep}(\lambda)(r) \circ (\text{Rep(S)}(r))^m \circ \text{Rep}(\lambda)(r)$ by Proposition 2.9.

$= \text{Rep}(\lambda)(r) \circ S^m \circ \text{Rep}(\lambda)(r) \circ S^m \circ \text{Rep}(\lambda)(r)$ $\forall\ r \in J$.

Thus $\text{Rep}(\lambda)(r) \in C_{mb}\ \forall\ r \in J$.

Theorem 5.4. The class C_{mq} of all m-fuzzy bi-ideals of a ternary semiring S is projection closed.

Proof. Let $\lambda \in C_{mq}$. Then $\lambda \in C_{ss}$ and $\lambda \circ S^m \cap S^m \lambda \circ S^m \cap S^m \circ \lambda \subseteq \lambda$. Utilizing Theorem 3.4, it is required to show that $\text{Rep}(\lambda)(r) \in C_{mq}\ \forall\ r \in J$. For $r \in J$, we have,

$\text{Rep}(\lambda)(r) \geq \text{Rep}(\lambda \circ S^m \cap S^m \circ \lambda \circ S^m \cap S^m \circ \lambda)(r)$ by Proposition 2.8.

$=\text{Rep}(\lambda \circ S^m)(r) \cap \text{Rep}(S^m \circ \lambda \circ S^m)(r) \cap \text{Rep}(S^m \circ \lambda)(r)$ by Proposition 2.7.

$=\text{Rep}(\lambda)(r) \circ (\text{Rep(S)}(r))^m \cap (\text{Rep(S)}(r))^m \circ \text{Rep}(\lambda)(r) \circ (\text{Rep(S)}(r))^m \cap (\text{Rep(S)}(r))^m \circ \text{Rep}(\lambda)(r)$ .by Proposition 2.9.

$=\text{Rep}(\lambda)(r) \circ S^m \cap S^m \circ \text{Rep}(\lambda)(r) \circ S^m \cap S^m \circ \text{Rep}(\lambda)(r)$

Thus $\text{Rep}(\lambda)(r) \in C_{mq}\ \forall\ r \in J$.

Theorem 5.5. $C_{m,\,tilt}$, $C_{m,\,til}$, $C_{m,\,tir}$ and $C_{m,\,ti}$ are projection closed.

Proof. Let $\lambda \in C_{m,\,tilt}$. Then $\lambda \in C_{ss}$ and $\lambda \circ \lambda \circ S^m \circ \lambda \circ S^m \circ \lambda \circ \lambda \subseteq \lambda$. Utilizing Theorem 3.4, it is required to show that $\text{Rep}(\lambda)(r) \in C_{m,\,tilt}\ \forall\ r \in J$. For $r \in J$, we have,

$\text{Rep}(\lambda)(r) \geq \text{Rep}(\lambda \circ \lambda \circ S^m \circ \lambda \circ S^m \circ \lambda \circ \lambda)(r)$ by Proposition 2.8

$=\text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r) \circ (\text{Rep(S)}(r))^m \circ \text{Rep}(\lambda)(r) \circ (\text{Rep(S)}(r))^m \circ \text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r)$ by Proposition 2.9

$=\text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r) \circ S^m \circ \text{Rep}(\lambda)(r) \circ S^m \circ \text{Rep}(\lambda)(r) \circ \text{Rep}(\lambda)(r)$.

Thus, $\text{Rep}(\lambda)(r) \in C_{m,\,tilt}\ \forall\ r \in J$.

Theorem 5.6. Let $m \geq 1$. Then, following statements hold in a ternary semiring S:

(1) Every fuzzy left (lateral, right) tri-ideal of S is also a m-fuzzy left (lateral, right) tri-ideal of S.

(2) Every m-fuzzy bi (quasi) -ideal of S is also a m-fuzzy tri-ideal of S.

Proof. By Theorems 3.3 - 5.5, the classes C_{til}, C_{tilt}, C_{tir}, C_{ti}, $C_{m,\,til}$, $C_{m,\,tilt}$, $C_{m,\,tir}$, $C_{m,\,ti}$, C_{mb}, and C_{mq} are projection closed. Therefore, the result of this theorem follows in view of Proposition 2.11, Proposition 2.5 and Theorem 4.1, 4.2 and 4.6 of [10] by proceeding similar to Theorem 4.1.

Theorem 5.7. If $\lambda \in C_{m,ti}$ and $g, h \in C_{ss}$, *then* $\lambda \circ g \circ h$, $g \circ \lambda \circ h$, $g \circ h \circ \lambda \in C_{m,ti}$.

Proof. We establish that if $\lambda \in C_{m,\,ti}$ and $\mu, \delta \in C_{ss}$, *then* $\lambda \circ g \circ h \in C_{m,\,ti}$. Define the classes $C_A = \{T_1\, \chi_{K_1}\, \chi_{K_2} : T_1 \in C_s$ *and* K_1, K_2 are subsemirings of S$\}$ and $C_A = \{\lambda \circ g \circ h : \lambda \in C_{m,\,ti},\ g, h \in C_{ss}\}$ be crisp and fuzzy classes defined on S. By proceeding as in Theorem 5.3, it can be established easily that C_A is projection closed. Also, by Theorem 5.5, $C_{m,\,ti}$ is projection closed. Therefore, by Proposition 2.11, $C_A \subseteq C_{m,\,\text{ti}}$ if and only if $C_A \subseteq C_{m,\,\text{ti}}$. The later proposition follows in view of Theorem 4.4 of [10] and $P(S) \cong C(S)$ under the isomorphism *Chi*.

Theorem 5.8. The product of atleast three m-fuzzy tri-ideals of a ternary semiring S is also a m- fuzzy tri-ideal of S.

Proof. Define classes, $C_D = \{T_1 T_2 T_3 : T_1, T_2, T_3 \in C_{m,ti}\}$ and $C_D = \{\beta \circ \gamma \circ \delta : \beta, \gamma, \delta \in$

$C_{m,ti}\}$ in S. By proceeding as in Theorem 5.3 by using Proposition 2.9, it can be shown easily that C_D is projection closed. Since $C_{m,ti}$ is projection closed by Theorem 5.5, therefore, $C_D \subseteq C_{m,ti}$ if and only if $C_D \subseteq C_{m,ti}$ by Proposition 2.11. By Theorem 4.3 of [10], the product of atleast three m-tri-ideals of a semiring S is also a m-tri-ideal of S and $P(S) \cong C(S)$ under the isomorphism *Chi*, we get, $C_D \subseteq C_{m,ti}$. Hence $C_D \subseteq C_{m,ti}$.

Analogously the subsequent theorem that extends Theorem 4.8 of [10] to fuzzy setting.

Theorem 5.9. Every l-fuzzy left ideal, m-fuzzy lateral ideal and n-fuzzy right ideal of a ternary semiring with multiplicative identity is a l-fuzzy left tri-ideal, m-fuzzy lateral tri-ideal and n-fuzzy right tri-ideal respectively.

Theorem 5.10. If $\beta \in C_{l,til}$, $\gamma \in C_{n,tir}$ and $\delta \in C_{m,tilt}$. Then $\beta \cap \gamma \cap \delta \in C_{k,ti}$, where k=max (l, n, m).

Proof. Define the classes $C_{l,n,m} = \{T_1T_2T_3 : T_1 \in C_{l,til}, T_2 \in C_{n,tir}, T_3 \in C_{m,tilt}$ and k=max(l,n,m)$\}$ and $C_{l,n,m} = \{\beta \circ \gamma \circ \delta : \beta \in C_{l,til}, \gamma \in C_{n,tir}, \delta \in C_{m,tilt}$ and k=max(l,n,m)$\}$ be the classes defined on S. It can be seen easily that $C_{l,til}, C_{n,tir}$ and $C_{m,tilt}$ are projection closed. Similar to Theorem 4.3, it can be shown easily that $C_{l,n,m}$ is projection closed. The result follows in view of $P(S) \cong C(S)$ under the isomorphism *Chi*, Proposition 2.11 and Theorem 4.9 of [10].

Theorem 5.11. If λ be a m-fuzzy tri-ideal of ternary semiring S and β be a m-fuzzy tri-ideal of λ such that $\beta \circ \beta \circ \beta = \beta$. Then β is a m-fuzzy tri-ideal of S.

Proof. Define the classes:

$C = \{g_1 \in C'_{m,\text{ ti}} : g_1 \circ g_1 \circ g_1 = g_1\}$ and

$C = \{\beta \in C'_{m,\text{ ti}} : \beta \circ \beta \circ \beta = \beta\}$, where $C'_{m,\text{ ti}}$ and $C'_{m,\text{ ti}}$ are crisp and fuzzy classes of m-tri ideals of λ.

It can be show easily that $C'_{m,\text{ ti}}$ is projection closed.

The result follows in view of Proposition 2.11, Corollary 4.1 of [10] and $P(S) \cong C(S)$ under the isomorphism *Chi*.

Analogously the subsequent theorem extends Theorem 4.10 of [10] to fuzzy setting.

Theorem 5.12. If λ be a m-fuzzy left (right, lateral) tri-ideal of ternary semiring S and β be a m-fuzzy left (right, lateral) tri-ideal of λ such that $\beta \circ \beta \circ \beta = \beta$. Then λ is a m-fuzzy left (right, lateral) tri-ideal of S.

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